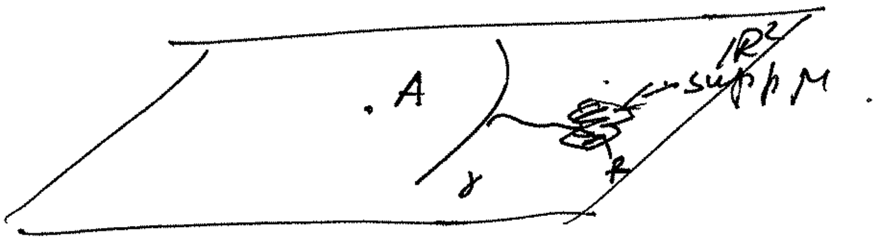
<u>Open Problem</u> (Generalized Antenna Problem)

Does there exist a compactly supported measure μ, $\operatorname{supp}\mu \cap \gamma = \emptyset$, s.t.

$$u(A) = \int u \, d\mu \quad \text{for all}$$

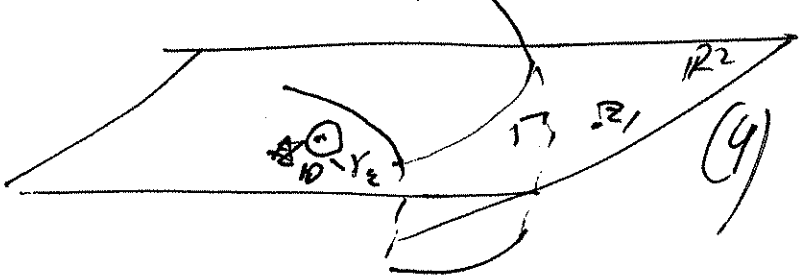
$$u\big|_\gamma = 0, \quad \Delta u + \lambda^2 u = 0 \quad \text{near } \gamma.$$

In other words can one "cut-off the umbilic cord" connecting B to γ.



(VI) The SRL and Huygens' Principle

Let us return to 2 dimensions and sketch yet another view point of the SRP blended from ideas of P. Garabedian and H. Lewy.



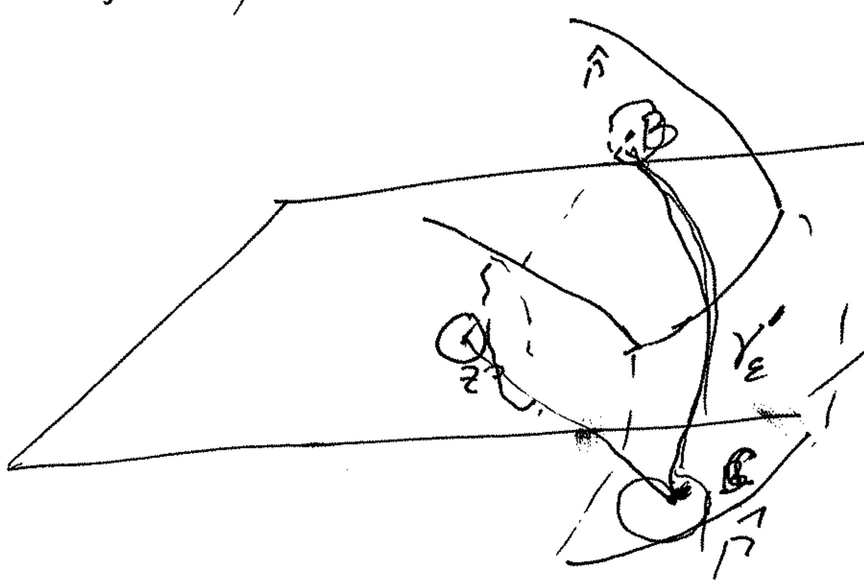
(4) $$u(z^0) = \frac{1}{2\pi}\int_{\gamma_\varepsilon}\left[u(\zeta)\frac{\partial}{\partial n_\zeta}\log|z^0-\zeta| - \frac{\partial u}{\partial n_\zeta}\log|z^0-\zeta|\right] ds_\zeta$$

where $\gamma_\varepsilon = \{ z : |z - z^0| = \varepsilon \}$ is a small circle surrounding z^0, n_ξ is the outward normal to it and ds_ξ is an arclength.

The form we are integrating in (*) is closed with a logarithmic singularity on the cone

$$K_{z^0} = \{ (z - z^0)(w - \bar{z}^0) = 0 \}$$

in $\mathbb{C}^2$. Therefore, we can move γ_ε homotopically ~~safely~~ to $\hat{\Gamma}$ avoiding K_{z^0} without changing the value of the integral in (*) until we lay it on $\hat{\Gamma}$ surrounding two points $\{B, C\} = K_{z^0} \cap \hat{\Gamma}$.



Since $u|_{\hat{\Gamma}} = 0$ the integral becomes

$$u(z^0) = \frac{1}{4\pi i} \int_{\gamma_\varepsilon'} \log (z - z^0)(w - \bar{z}^0) \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial w} \right.$$

where γ_ε' consists of two small circles around B, C joined by a "handle" on $\hat{\Gamma}$.
The integrals over the two small circles tend to zero when $\varepsilon \downarrow 0$ while the integral over the handle travelled twice tends to

$$u(z^0) = \frac{1}{2} \int_B^C \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial w} dw \right).$$

Since the logarithm on one side of the handle differs by that on the other side by $2\pi i$

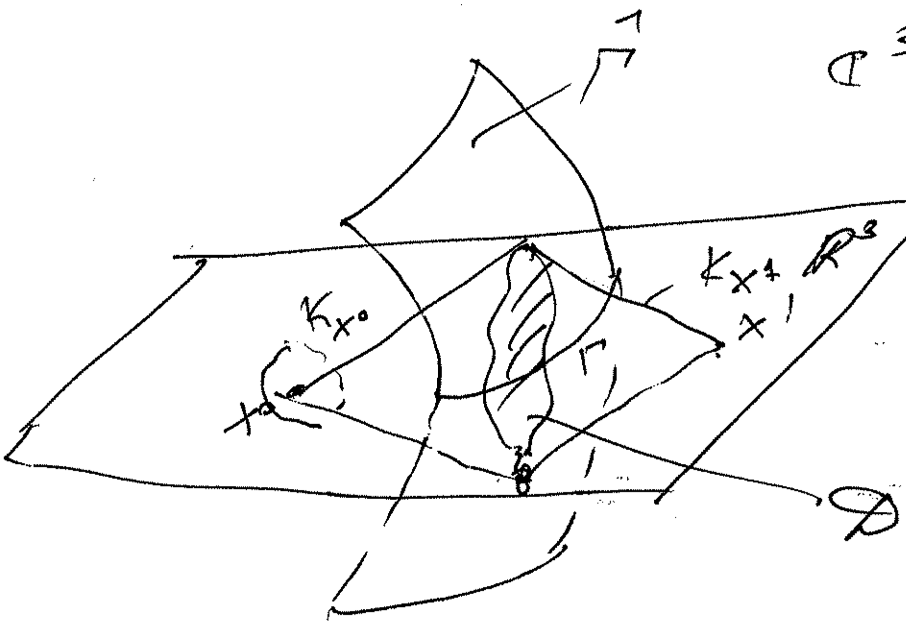
Applying the same argument to z^1 we obtain the same representation with the change of direction on BC, hence

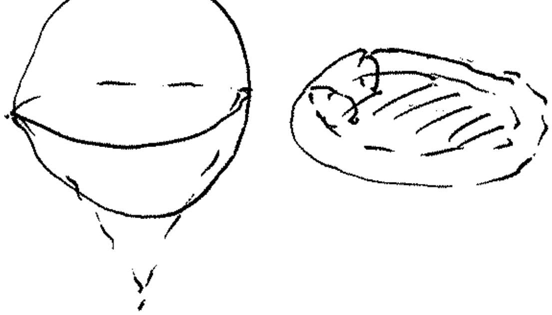
$$u(z^0) = -u(z^1), \quad \#.$$

Now if we follow that process in 3 dimensions, or more generally in any $(2n+1)$-dimensional space the logarithmic kernel has to be replaced by the newtonian kernel

$$N(x,y) = c_N \, |x-y|^{2-n}$$

that has a square root-like multivalued singularity around the isotropic cone,





Hence when we move the S^{2n}-sphere in $\mathbb{C}^{2n+1}$ and flatten it onto $\hat{\Gamma}$ we'll end up with integrating twice over the 2n-dim domain $\mathcal{D}$ on $\hat{\Gamma}$. The form we are integrating will be of the form

$$u(x^0)=\int_{\mathcal{D}} N(z, x^0)\,(\ldots)$$

2n-dim form involving grad u,

Where $(\ldots)$ can be (by the Cauchy-Kovalevskay theorem) made rather arbitrary.

Hence, if (RL)

~~iff~~ $u(x^0)+Ku(x')=0$ (RL)

$\Delta u=0,\ u|_{\Gamma}=0$

holds for x^0, x' we must have,

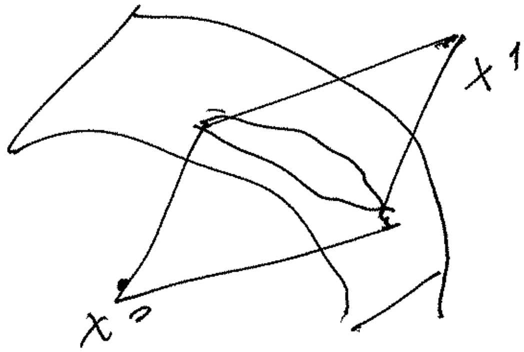
$$N(\cdot, x^0) \mp K N(\cdot, x^1) \equiv 0 \text{ on } \hat{\Gamma}$$

which yields

<u>theorem</u> ('96, Ebenfelt-K.) If in an odd-dimensional space (RL) holds for two points sufficiently close to a hypersurface Γ, Γ must be either a sphere, or a plane.

Now, finally, let us discuss the even-dimensional case.

<u>Definition</u> Two points $x^0, x^1 \in R^n \setminus \Gamma$ are said to be SR points (Study reflection points) wrt Γ if $K_{x^0} \cap \hat{\Gamma} = K_{x^1} \cap \hat{\Gamma}$, $\hat{\Gamma}$ = complexification of Γ and $K_{x^0} = \{\sum_1^N (x_i - x_i^0)^2 = 0\}$ denotes an isotropic cone with a vertex at x^0.



Intuitively (thinking in $\mathbb{R}^n$) if instead of isotropic cones we consider "light cones" $\{x : x_n^2 - \sum_1^{n-1} x_i^2 = 0\}$ (SR) in that context means that simply the cones with vertices at x^0, x^1 meet on Γ.

Now to get a glimpse into what we can expect in even dimensions let us "cheat" and assume (which need not be the case) that our surface $\hat{\Gamma}$ "appears" as a real hypersurface $\tilde{\Gamma}$ in the real subspace $W = i\mathbb{R}^3 \times \mathbb{R} = \{(ix', x_4)\}$ (in general, this intersection has codimension 2).

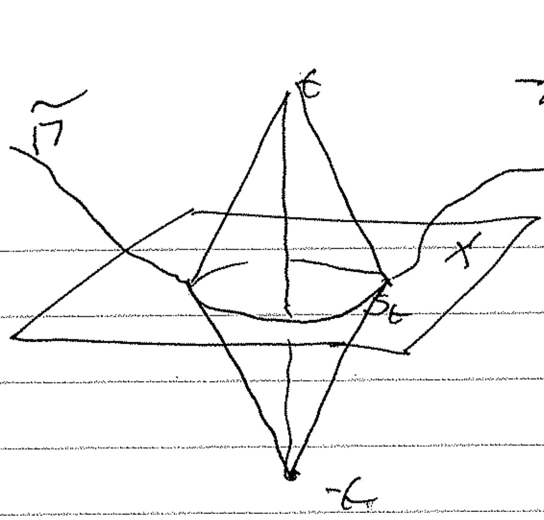
Let $x^0 = (0,0,0,t)$ and $x' = (0,0,0,-t)$ be SR points. Obviously, we have $K_{x^0} \cap K_{x'} \subset \{z_4 = 0\}$

Any $u \in H_0(\Gamma)$ satisfies in W the wave equation

$$\sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2} - \frac{\partial^2 u}{\partial x_4^2} = 0.$$

$K_{x^0} \cap W$, $K_{x^1} \cap W$, are the ordinary "light cones" emanating from x^0 and x' and they meet on $\tilde{\Gamma}$ along 2-dimensional sphere

$$S_t = \{(ix', 0) : |x'| = t\}.$$



Now, by Kirchoff's formula the value of u at x^0, say can be calculated in terms of the Cauchy data on the plane $x_4 = 0$ along the sphere S_t only. ($u = 0$ on S_t!)

$$u(x^0) = \frac{c}{t^2}\int_{S_t} \left(t\frac{\partial u}{\partial x_4} + \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} x_j \right) dS. \quad (5)$$

If we denote by $g_{x^0}(x) = -\sum_{j=1}^{3} (x_j - x_j^0)^2 + (x_4 - x_4^0)^2$ the defining function of the isotropic cone $-\nabla g_{x^0} = 2[(x - x^0), -(x_4 - x_4^0)] = 2(x, t)$ on the plane $\{x_4 = 0\}$, we can rewrite (5) in the form

$$u(x^0) = -\frac{c}{2t^2}\int_{S_t} \nabla u \cdot \nabla g_{x^0}\, dS =$$

$\nabla u \parallel \vec{n}$ to $\tilde{\Gamma}$! $u|_{\tilde{\Gamma}} = 0$

$$= -\frac{c}{2t^2}\int_{S_t} \frac{\partial u}{\partial n} (\vec{n} \cdot \nabla g_{x^0})\, dS. \quad (6)$$

where $\vec{n}$ denotes the unit normal to $\tilde{\Gamma}$. Similarly

$$u(x^1) = -\frac{c}{2t^2}\int_{S_t} \frac{\partial u}{\partial n}\left(\vec{n}\cdot\nabla g_{x^1}\right) dS \tag{7}$$

Now if (RL) $u(x^0) + K\,u(x^1) = 0$ for all u satisfying the equation and vanishing on Γ, then since $\frac{\partial u}{\partial n}$ can be more or less arbitrary, (RL) must yield.

$$\left(\nabla g_{x^0} + K\,\nabla g_{x^1}\right)\cdot\vec{n} = 0 \text{ on } S_t \tag{8}$$

To fix the ideas assume that $\tilde{\Gamma}$ can be written as $x_4 = \varphi(x' = (x_1, x_2, x_3))$. Then, since $\tilde{\Gamma} \cap \{x_4 = 0\} \supset S_t$, so on S_t $\varphi = 0$, hence

$\nabla\varphi = \Psi(x')x'$ on S_t.

$\nabla g_{x^0} = -2(x', t)$ on S_t $\qquad \vec{n} \parallel (\Psi(x')x', 1)$.

$\nabla g_{x^1} = \cancel{+}2(\cancel{x}', -t)$ on S_t. $\qquad |x'| = t$ on St

Thus (8) implies $\qquad \left((x', t) + K(x', -t), (\Psi(x')x', -1)\right) = 0$.

$(1+K)\Psi(x')t^2 \cancel{\;} + t(1-K)\cancel{\;} = 0$

Thus, $\varphi(x') = \frac{1}{t}\frac{1-K}{1+K} =$ const on S_t.

Now, calculating that on $\tilde{\Gamma}$ we have

$$-dg_{x^0} = 2(x', t)(dx' + \nabla\varphi\, dx') =$$

$$= 2(x' + t\nabla\varphi)\cdot dx'$$

$$-dg_{x^1} = 2(x' - t\nabla\varphi_1)\cdot dx'$$

and substituting

$$\nabla\varphi = \frac{1}{t}\frac{1-k}{1+K}x'$$

we conclude that on $\tilde{\Gamma}$

$$dg_{x^0} = \frac{1}{K}dg_{x^1} \qquad (9)$$

We shall call (9) the SSR, the Strong Study Relation. We assume that (9) holds together with (SR).

It is not too difficult to show that

that (SSR) in $\mathbb{R}^4$ is equivalent to saying that infinitesimally (up to order) along the intersection $K_{x^0} \cap \hat{\Gamma}$ ($= K_{x^1} \cap \hat{\Gamma}$) the surface $\hat{\Gamma}$ is axially symmetric with the axis of symmetry.

For even $n \geq 6$ we say that (SSR) holds ~~if~~ for points x^0, x^1 if (SR) holds and (S) holds on $\hat{\Gamma}$ up to the order $p = \frac{n-2}{2}$ (1 for $n=4$!). Loosely speaking, at the intersection $K_{x^0} \cap \hat{\Gamma} = K_{x^1} \cap \hat{\Gamma}$, $\hat{\Gamma}$ is tangent to a sphere (or a plane) along that intersection up to the order $n-2/2$.

The following theorem (Ebenfelt-K, '96)

completely settles the question of (RL) in even dimensions

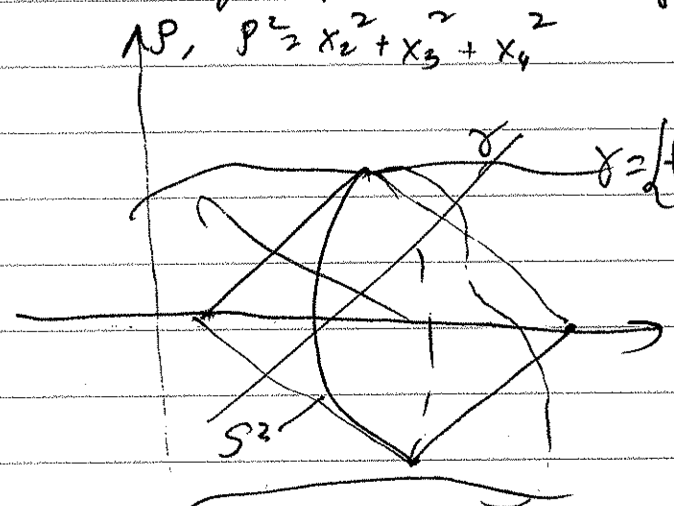
<u>Theorem</u> If n is even, then (RL)

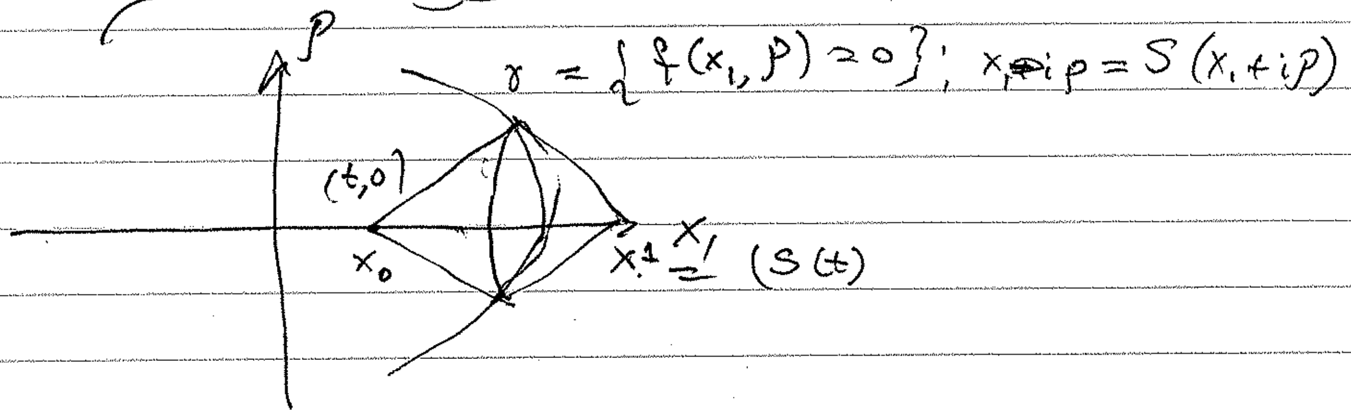
$$u(X^0) + K u(X^1) = 0, \quad u \in Har_0(\Gamma)$$

holds iff X^0, X^1 satisfy (SRR).

<u>Examples</u> $n = 4$

(i) Axially symmetric surfaces in $\mathbb{R}^4$.

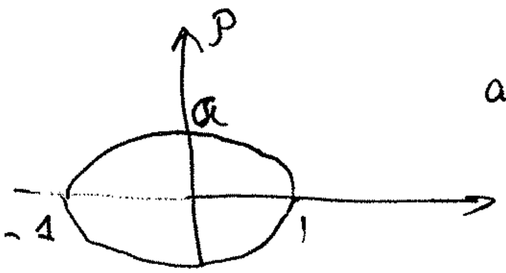




Then, (SR), (SSR) hold with $\lambda = -\frac{1}{S'(t)}$.
so $K = -S'(t)$.

(ii) Let $\gamma \subset R^2$ be the ellipse

$$x_1^2 + \frac{\rho^2}{a^2} = 1 \quad , \quad a^2 \neq 1$$

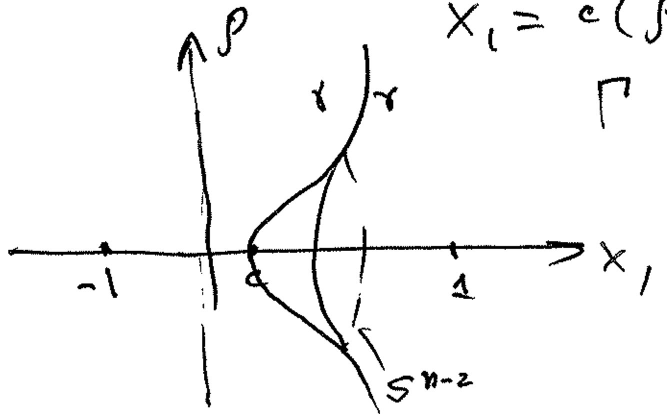


and Γ be the axially symmetric ellipsoidal surface in $\mathbb{R}^n$, n even, ≥ 4.

Since $\hat{\Gamma}$ is a quadric, it cannot have an order of contact greater than 1 with another quadric unless the two are the same. Hence, since $a^2 \neq 1$, no two points satisfy SSR with respect to Γ when $n \geq 6$, even and reflection always fails.

(iii) Let $\gamma \subset \mathbb{R}^2$ be the curve

$$x_1 = c(\rho^2+1)^k \quad , \quad k \geq 2 \quad , \quad c > 0 \text{ small}$$



Γ be the corresponding surface of revolution in $\mathbb{R}^n$, $n \geq 4$, even.

The points $x^0 = (-1, 0 \ldots, 0)$, $x^1 = (1, 0 \ldots 0)$, $\lambda = 1$
satisfy SSR iff $n \leq 2k$.
$(n=4, K \geq 2)$ $n=6$, $K \geq 3$ etc.

References.

(1) D. Khavinson & H. S. Shapiro Remarks on the reflection principle for harmonic functions J. d'Analyse Math. 54 (1991), 60-76

[2]. D. Khavinson, On reflection of harmonic functions in surfaces of revolution, Complex Variables Appl. 17 (1991), 7-14

[3] P. Ebenfelt, D. Khavinson, On point-to-point reflection of harmonic function across real-analytic hypersurfaces in R^n J. d'Analyse Math. 68 (1996), 145-18.