Open Problem (Generalized Antenna Problem)

Does there exist a compactly supported measure $\mu$, supp $\mu \cap \Gamma = \emptyset$, s.t.

\[ u(A) = \int_{A} d\mu \quad \text{for all} \]

\[ u \big |_{\Gamma} = 0, \quad \partial u + \lambda^2 u = 0 \quad \text{near} \quad \Gamma. \]

In other words one can "cut-off the umbilic cord" connecting $B$ to $A$.

(VI) The SRL and Huygen's Principle

Let us return to 2 dimensions and sketch yet another viewpoint of the SRLP blended from ideas of P. Garabedian and H. Lewy.
where $\delta_2 = \{ z : |z - z_0| < \delta \}$ is a small circle surrounding $z_0$. $n_2$ is the outward normal to it, and $ds_2$ is an arc length.

The form we are integrating in (a) is closed with a logarithmic singularity on the cone

$$K_{z_0} = \{ (z-z_0)(w-z_0) = 0 \}$$

in $\sigma^2$. Therefore, we can move $z_2$ homotopically to $\tilde{z}$ avoiding $K_{z_0}$ without changing the value of the integral in (b) until we lay it on $\tilde{z}$ surrounding two points.

Since $u|_{\tilde{z}} = 0$, the integral becomes

$$u(z_0) = \frac{1}{4\pi i} \oint_{\delta_2} \log(z-z_0)(w-z_0) \left( \frac{\partial u}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right).$$
where \( \mathcal{K} \) consists of two small circles around \( B, C \) joined by a "handle" on \( F \).

The integrals over the two small circles tend to zero when \( \epsilon \to 0 \) while the integral over the handle travelled tends to
\[
\frac{1}{2} \int_{\mathcal{B}} \left( \frac{2u}{r^2} \frac{I_2}{2} - \frac{2u}{r} I_4 \right)
\]

Since the logarithm on one side of the handle differs by that on the other side by \( 2\pi i \).

Applying the same argument to \( 2^+ \) we obtain the same representation with the change of direction on \( B, C \), hence
\[
u(z^0) = -\nu(z^1), \quad \#.
\]

Now if we follow that process in 3 dimensions, or more generally in any \((2n+1)\)-dimensional space the logarithmic kernel has to be replaced by the Newtonian kernel
\[
N(x, y) = CN \frac{1}{|x - y|^{2-n}}
\]
that has a square root-like multivalued singularity around the isotropic cone.
Hence when we move the semi-sphere in $C^{2n+1}$ and flatten it onto $\hat{\Gamma}$ we'll end up with integrating twice over the domain $\mathcal{D}$ on $\hat{\Gamma}$. The form we are integrating will be of the form

$$\dot{\omega}(x^0) = \int N(z, x^0) \left( \ldots \right) \, d\mathcal{D}$$

2n-dim form involving grad $u$,

where (\ldots) can be (by the Cauchy – Kowalevskaya theorem) made rather arbitrary.

Hence, if (RL)

$$u(x^0) + Ku(x') = 0 \quad \text{(RL)}$$

$$\Delta u = 0, \quad u|_{\Gamma} = 0$$

holds for $x^0, x'$ we must have
\[ N(\cdot, x^0) + K N(\cdot, x') = 0 \text{ on } \hat{\Gamma} \]

which yields theorem (\text{\cite{Eberle}}). If in an odd-dimensional space (RH) holds for two points sufficiently close to a hypersurface \( \Gamma \), \( \Gamma \) must be either a sphere, or a plane.

Now, finally, let us discuss the even-dimensional case.

**Definition.** Two points \( x^0, x^1 \in \mathbb{R}^n \setminus \Gamma \) are said to be SR points (Study reflection points) with \( \Gamma \) if \( K_{x^0} \cap \hat{\Gamma} = K_{x^1} \cap \hat{\Gamma} \), \( \hat{\Gamma} \) complexification of \( \Gamma \) and \( K_{x^0} = \frac{1}{2} \sum_i (x_i - x^0_i)^2 = 0 \) denotes an isotropic cone with a vertex at \( x^0 \).

Intuitively (thinking in \( \mathbb{R}^n \)) if instead of isotropic cones we consider "light cones" \( \| x \| = x_n - \frac{1}{2} x_i^2 = 0 \), (SR) in that context means that simply the cones with vertices at \( x^0, x^1 \) meet on \( \Gamma \).
Now to get a glimpse into what we can expect in even dimensions let us "cheat" and assume (which need not be the case) that our surface $\Gamma$ appears as a real hypersurface $\Gamma$ in the real subspace $W = iR \times iR = \{(x, y)\}$ (in general, this intersection has codimension 2).

Let $x^0 = (0, 0, 0, t)$ and $x^1 = (0, 0, 0, -t) \in \mathbb{R}^6$.

Points. Obviously, we have $K_{x^0} \cap K_{x^1} \subset \{(x, y) : y = 0\}$.

Any $u \in H^0(\Gamma)$ satisfies in $W$ the wave equation

$$\sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2} - \frac{\partial^2 u}{\partial x_4^2} = 0.$$ 

$K_{x^0} \cap W, K_{x^1} \cap W$ are the ordinary light cones emanating from $x^0$ and $x^1$ and they meet on $\Gamma$ along 2-dimensional spheres

$$S_t = \{(ix, 0) : |x| = t\}.$$
Now, by Kirchhoff's formula, the value of $u$ at time $t$ can be calculated in terms of the Cauchy data on the plane $x_4 = 0$ along the sphere $S_t$ only. $(u = 0$ on $S_t)$

$$u(x^0) = \frac{C}{t^2} \int_{S_t} \left( t \frac{\partial u}{\partial x_4} + \frac{3}{2} \frac{\partial u}{\partial x_j} \right) dS.$$ \hspace{1cm} (57)

If we denote by $g_{x^0}(x) = \frac{2}{x^4} (x_1 - x_1^0) + (x_4 - x_4^0)^2$ the defining function of the isotropic cone $-\nabla g_{x^0} = \partial \left[ (x - x^0)^2 (x_4 - x_4^0)^2 \right] = 2 (x', t)$ on the plane $x_4 = 0$, we can rewrite (5) in the form

$$u(x^0) = \frac{C}{2t^2} \int_{\partial S_t} \nabla u \cdot \nabla g_{x^0} dS =$$

$$\int_{\partial S_t} \frac{C}{2t^2} \frac{\partial g_{x^0}}{\partial n} (n \cdot \nabla g_{x^0}) dS \hspace{1cm} (6)$$
where \( \mathbf{n} \) denotes the unit normal to \( \tilde{\Gamma} \). Similarly

\[
\bar{u}(x') = -\frac{c}{2t^2} \int_{S_\tilde{\Gamma}} \frac{\partial u}{\partial n} \cdot \mathbf{n} \cdot \bar{v}_{x_2} \, dS.
\]

(7)

Now if (7) \( u(x^0) + Ku(x') = 0 \) for all \( u \) satisfying the equation and vanishing on \( \tilde{\Gamma} \), then since \( \frac{\partial u}{\partial n} \) can be more or less arbitrary \( (7) \) must yield

\[
\left( \frac{\partial \bar{v}_{x_0}}{\partial x_0} + K \frac{\partial v_{x_1}}{\partial x_1} \right) \cdot \mathbf{n} = 0 \text{ on } S_\tilde{\Gamma}.
\]

(8)

To fix the ideas, assume that \( \tilde{\Gamma} \) can be written as \( x_4 = \psi(x', 0) \). Then, since \( \tilde{\Gamma} \cap \{ x_4 = 0 \} \supset S_\tilde{\Gamma} \), so on \( S_\tilde{\Gamma} \) \( \psi = 0 \), hence

\[
\nabla \psi = \psi(x') \nabla x' \text{ on } S_\tilde{\Gamma}.
\]

\[
\frac{\partial \psi}{\partial x_0} = -2(x', t) \text{ on } S_\tilde{\Gamma} \quad \Rightarrow \quad (\psi(x')x'_{x_0}, 1) \quad (x', t) \text{ on } S_\tilde{\Gamma}.
\]

\[
\frac{\partial \psi}{\partial x_1} = K \frac{\partial x'}{\partial x_1} \text{ on } S_\tilde{\Gamma} \quad \Rightarrow \quad (x', t) + K (x', t) \left( \frac{\partial x'}{\partial x_1} \right) = 0.
\]

Thus (8) implies

\[
(\psi(x')x'_{x_1} + (x', t) \nabla x'_{x_1}) = 0.
\]
Thus, \( Y(x') = \frac{1}{t} \frac{1 - K}{t + 1/K} = \text{const on } \mathbb{S}^1 \).

Now, calculating that on \( \mathbb{S}^1 \), we have

\[-dg_{x_0} = 2(x'_1 t + \varphi \, dx'_1) = 2(x'_1 + \varphi) \cdot dx'_1 \]
\[-dg_{x_1} = 2(x'_1 + \varphi), \, dx'_1 \]

and substituting

\[\begin{align*}
\varphi &= \frac{1}{1 + K} x'_1 \\
Y &= \frac{1}{t} \
\end{align*}\]

we conclude that on \( \mathbb{S}^1 \)

\[dg_{x_0} = \frac{1}{K} \frac{1 - K}{t + 1/K} \, dg_{x_1} \quad (9) \]

We shall call (9) the SSR, the strong study relation. We assume that (9) holds together with (SR).

It is not too difficult to show that
That (SSR) in $\mathbb{R}^4$ is equivalent to saying that infinitesimally (up to order 1) along the intersection $K_{x_0} \cap \hat{T} = K_{x_1} \cap \hat{T}$ the surface $\hat{T}$ is axially symmetric with the axis of symmetry.

For even $n \geq 6$ we say that (SSR) holds if for points $x^0, x^1$ if (SSR) holds and (3) holds on $\hat{T}$ up to the order $p = \frac{n-2}{2}$ (1 for $n=4$!). Loosely speaking, at the intersection $K_{x_0} \cap \hat{T} = K_{x_1} \cap \hat{T}$, $\hat{T}$ is tangent to a sphere (or a plane) along that intersection up to the order $n-2/2$.

The following theorem (Ebenfelt-K, '96)
completely settles the question of (RL) in even dimensions.

**Theorem** If \( n \) is even, then (RL)

\[
u(x^0) + ku(x^1) = 0, \quad u \in H_{2\alpha} (\Gamma)
\]

holds iff \( x^0, x^1 \) satisfy (SRR).

**Examples** \( n = 4 \)

(1) Axially symmetric surfaces in \( \mathbb{R}^4 \)

\[
\Phi, \quad p = x_2 x_3 + x_4^2
\]

\[
\Phi \equiv \{ f(x_1, \hat{p}) = 0 \}
\]

\[
\delta = \{ f(x_1, p) = 0 \}, \quad x \times p \equiv S(x_1 + ip)
\]

(\( \xi, 0 \))

\[
x_0 \quad x_1 \text{ at } (S(\xi))
\]
Then, \((SR),(SSR)\) hold with \(k = -\frac{1}{S'(t)}\).

so \(k = -S'(t)\).

(ii) Let \(y \in \mathbb{R}^2\) be the ellipse

\[ x_1^2 + \frac{p^2}{a^2} = 1, \quad a \neq 1 \]

and \(\Gamma\) be the axially symmetric ellipsoidal surface in \(\mathbb{R}^n\), \(n\) even, \(n \geq 4\).

Since \(\Gamma\) is a quadric, it cannot have an order of contact greater than 1 with another quadric unless the two are the same. Hence, since \(a^2 \neq 1\), no two points satisfy SSR with respect to \(\Gamma\) when \(n \geq 6\), even and reflection always fails.

(iii) Let \(y \in \mathbb{R}^2\) be the curve

\[ x_1 = c(p^2 + 1)^k, \quad k > 2, \quad c > 0 \text{ small} \]

\(\Gamma\) be the corresponding surface of revolution in \(\mathbb{R}^n\), \(n \geq 4\), even.
The points $x^0 = (-1, 0, \ldots, 0)$, $x^1 = (1, 0, \ldots, 0)$, $\lambda = 1$ satisfy $SSR$ iff $n \leq 2k$.

$(n=4, k \geq 2)$: $n = 6, k \geq 3$ etc.

\textbf{References.}

