Summary of the one-hour tutorial
Introduction to the Painlevé property, test and analysis

Robert Conte
Centre de mathématiques et de leurs applications, École normale supérieure de Cachan
et CEA/DAM, France.

Micheline Musette
Dienst Theorectische Natuurkunde, Vrije Universiteit Brussel, Belgium.

Summary. In this short time, only the essential features can be presented. For convenience
of the audience, they are outlined below. Full details can be found in The Painlevé handbook

1 Motivation. Definition of the Painlevé property

The goal is, given some algebraic ordinary differential equation (ODE), to
find a closed form expression for its general solution.

Fact: whenever one succeeds to find the general solution $u(x)$ of an ODE,
this $u$ only involves solutions of other, usually lower order ODEs. For instance,
$\Gamma(x)$ can never contribute. Hence the

Motivation (L. Fuchs, Poincaré): define new functions by ODEs.
Function $\iff$ singlevaluedness.

Linear ODEs are all able to define a function because their general solution
has no singularity which depends on the constants of integration.

What about nonlinear ODEs? The singularities of their solutions have two
properties: on one hand critical or noncritical (i.e. local multivaluedness or local
singlevaluedness), on the other hand fixed (their location does not depend on
the initial conditions) or movable (the opposite). See examples in Table 1.

For singlevaluedness, the only worry is: at the same time, movable and
critical. Hence the

Definition. Painlevé property (PP) of an ODE:= its general solution
has no movable critical singularities.
Table 1: The four configurations of singularities. The location $c$ depends on the constants of integration (=on the initial conditions)

<table>
<thead>
<tr>
<th>Singularity Type</th>
<th>critical</th>
<th>noncritical</th>
</tr>
</thead>
<tbody>
<tr>
<td>movable</td>
<td>$(x - c)^{1/2}$</td>
<td>$e^{1/(x-c)}$</td>
</tr>
<tr>
<td>fixed</td>
<td>$\log(x - 1)$</td>
<td>$(x - 2)^3$</td>
</tr>
</tbody>
</table>

Any other definition, such as “All solutions have only movable poles”, is incorrect, see Chazy (1911) and [5, FAQ section].

Warning: Essential singularities are not involved in the definition of the PP.

In order to find new functions, one must (i) investigate nonlinear ODEs of order one, then two, then three . . . ; (ii) select those which possess the PP, (iii) prove whether their general solution defines a new (or an old) function.

## 2 By-products of the definition of the PP

1. **Painlevé test**:=collection of algorithms (life is finite!) providing necessary conditions for the PP.

   Warning: These conditions are a priori not sufficient, i.e. passing the P test does not imply possessing the PP. Indeed, as an algorithm, the test must terminate after a finite number of steps. Example of Picard (1893): $u(x) = \wp(\lambda \log(x - c_1) + c_2, g_2, g_3)$ obeys (by the elimination of $c_1, c_2$) a second order ODE and is singlevalued iff $2\pi i \lambda$ is a period, a transcendental condition on $\lambda, g_2, g_3$ impossible to obtain in a finite number of algebraic steps.

2. **Painlevé analysis**:= (by extension) methods based on singularities, in order to generate any kind of closed-form result (particular solution, first integral, Darboux polynomial, Lax pair, etc). More in [4].

3. **Classifications**:= exhaustive lists of ODEs in a certain class (e.g. third order second degree) which have the PP. State of the art in [5, Appendix A]: Gambier, Chazy, Bureau, Exton, Martynov, Cosgrove.
4. **New functions.** The main difficulty is to prove the irreducibility (to a linear ODE or to a nonlinear, lower order ODE).

Order one: only one new function, the elliptic function, e.g. $\wp$.

Order two: only six, the Painlevé functions (Painlevé 1900, R. Fuchs 1905).

Order three: none.

Order four: at least five first degree ODEs (labeled F-V, F-VI, F-XVII, F-XVIII, Fif-IV in Cosgrove) have a singlevalued general solution which depends transcendentally on the four constants of integration. However, their irreducibility is unproven.

Example of a fourth order ODE with the PP, a transcendental dependence on the four constants of integration, and a reducible general solution: the ODE whose general solution is $u(x) = u_1(x) + u_2(x)$, where $u_j(x)$ obeys the first Painlevé equation. One must therefore be very cautious about the irreducibility.

## 3 The Painlevé test

The main message is that the P test does *not* reduce to the method of Kowalevski (1889) and Gambier (1910), popularized by Ablowtiz, Ramani and Segur (1977, 1978, 1980).

The main methods of the test are

1. **$\alpha$-method of Painlevé** (1900). This is the most powerful, but it involves solving differential equations, while all other methods involve solving algebraic equations. Tutorial presentation in [2, §5.5]

2. **Method of Kowalevski and Gambier**, later made rigorous by Bureau (1939, 1964). This well known method consists in requiring the existence, near every movable singularity, of a Laurent series able to represent the general solution. We will not detail it here.

Two errors should not be made:

*Error 1.* Discard negative integer Fuchs indices (also called resonance, or Kowalevski exponent) as containing no information. Example: the ODE

$$u''' + 3uu'' - 4u'^2 = 0, \quad (1)$$
admits the two families

\[ u \sim -60/(x-x_0)^2, \text{ Fuchs indices } -3, -2, -1, 20, \]  
\[ u \sim u_0/(x-x_0)^3, \text{ } u_0 \text{ arbitrary, Fuchs indices } -1, 0, \]

among which the first family has negative indices \(-3, -2\). See below.

Error 2. Interpret a negative integer Fuchs index as the presence of an essential singularity. Example: the ODE

\[ u'' + 3uu' + u^3 = 0 \]  

admits the family \( u \sim 2(x-x_0)^{-1} \) with the Fuchs indices \(-2, -1\), and it has no essential singularity because its general solution \( u = \frac{1}{x-a} + \frac{1}{x-b} \) has only poles.

There exist two situations making the method of Kowalevski and Gambier indecisive:

(a) presence of negative integers among the set of Fuchs indices,
(b) insufficient number of Fuchs indices, i.e. lower than the differential order of the ODE.

3. **Fuchsian perturbative method** [10, 3]. It deals with the first indecisive situation (negative integer Fuchs indices).

It considers the Laurent series of Gambier as the zero-th order of a Taylor series in a small parameter \( \varepsilon \) and requires singlevaluedness at each order in \( \varepsilon \). With the first family of (3), a movable logarithm arises from the Fuchs index \(-1\) at seventh order of perturbation.

4. **Non-Fuchsian perturbative method** [13]. This handles the second indecisive situation (number of Fuchs indices lower than the differential order), provided one knows a particular solution in closed form.

The Laurent series of the second family of (3) happens to terminate and defines the closed form two-parameter particular solution

\[ u^{(0)} = c(x-x_0)^{-3} - 60(x-x_0)^{-2}, \text{ } (c, x_0) \text{ arbitrary.} \]  

Under the perturbation \( u = u^{(0)} + \varepsilon u^{(1)} + \ldots \), the ODE for \( u^{(1)} \)

\[ E^{(1)} = E'(x, u^{(0)})u^{(1)} \equiv [\partial_x^4 + 3u^{(0)}\partial_x^2 - 8u_x^{(0)}\partial_x + 3u_{xx}^{(0)}]u^{(1)} = 0, \]
is known globally (not only locally near \( x = x_0 \)) because \( u^{(0)} \) is closed form, therefore one can test all its singularities for singlevaluedness. Testing \( x = x_0 \) (nonFuchsian) is what the Fuchsian perturbative method has done above. Testing \( x = \infty \) (Fuchsian) immediately uncovers a movable logarithm [13] arising from a Neumann function.

### 4 Methods for finding all elliptic solutions of a given ODE

Quite a number of presentations in this Tampa conference deal with finding, by various methods, some particular solutions of a given nonlinear ODE in the class of either elliptic functions or rational functions of one exponential or rational functions.

What we would like to point out is the existence of two methods, turned into algorithms, to find not only some but all the particular solutions in that class (elliptic and degenerate of elliptic).

The first method [14, 5, 6] implements a classical theorem of Briot and Bouquet, stating that any such solution obeys a first order algebraic ODE \( F(u', u) = 0 \) in which the degrees of the polynomial \( F \) in \( u \) and \( u' \) are known.

The second method [9] implements another classical result of Hermite, stating that any such solution is necessarily representable as the finite sum

\[
    u = C + P(x) + \sum_{k=1}^{N} \sum_{j=1}^{m_k} c_{k,j} \zeta^{(m)}(x - x_0 - a_k),
\]

in which \( x_0, C, c_{k,j}, a_k \) are constant \( \zeta \) is the function of Weierstrass, and \( P \) is a polynomial which is nonzero only for rational solutions.

Because they can find all such solutions, these two methods make obsolete all the previous methods.

### References


Russian translation: http://shop.rcd.ru/details/1304


math.CA, math.DS


